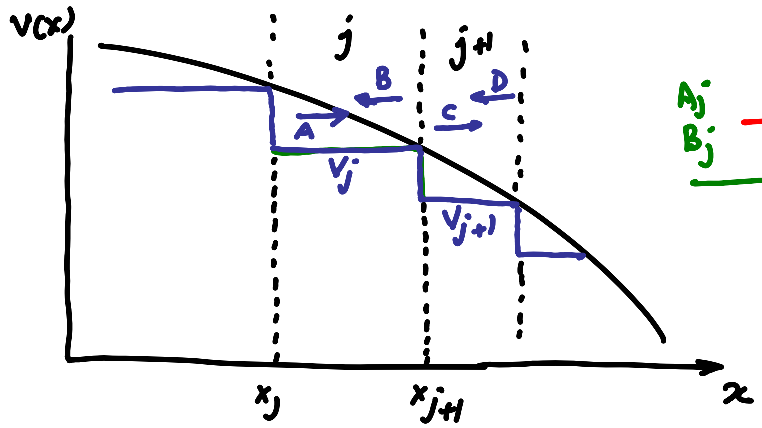
Transmission Matrix Method (TMM)

(The propagation matrix)

Note Title 2/4/2008



$A_j, B_j \rightarrow A_{j+1}, B_{j+1} \rightarrow C_{j+1}, D_{j+1}$

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \,?\, \begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}, \qquad ? \rightarrow \hat{P}_j$$

$$\begin{cases} \psi_j = A_{j+1} e^{ik_j x} + B_{j+1} e^{-ik_j x} \\ \psi_{j+1} = C_{j+1} e^{ik_{j+1} x} + D_{j+1} e^{ik_{j+1} x} \end{cases}$$

$$E - V_j = \frac{\hbar^2 k_j^2}{2m}$$

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note: V_j is potential energy. Sometimes V_j is voltage (electric potential). In that case potential energy for an electron in this voltage is: $-eV_j$

Apply B.C.:

$$\psi_j = \psi_{j+1}\Big|_{x_{j+1}} \Rightarrow A_{j+1} e^{ik_j x_{j+1}} + B_{j+1} e^{-ik_j x_{j+1}} = C_{j+1} e^{ik_{j+1} x_{j+1}} + D_{j+1} e^{-ik_{j+1} x_{j+1}}$$

$$\psi_j' = \psi_{j+1}'\Big|_{x_{j+1}} \Rightarrow A_{j+1}(ik_j) e^{ik_j x_{j+1}} + B_{j+1}(ik_j) e^{-ik_j x_{j+1}} = C_{j+1}(ik_{j+1}) e^{ik_{j+1} x_{j+1}} + D_{j+1}(ik_{j+1}) e^{-ik_{j+1} x_{j+1}}$$

Assume $x_{j+1} = 0$: $\Rightarrow$

$$\begin{cases} A_{j+1}+B_{j+1} = C_{j+1}+D_{j+1} \\ A_{j+1}-B_{j+1} = \dfrac{k_{j+1}}{k_j}C_{j+1} - \dfrac{k_{j+1}}{k_j}D_{j+1} \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \dfrac{k_{j+1}}{k_j} & -\dfrac{k_{j+1}}{k_j} \end{bmatrix}\begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}$$

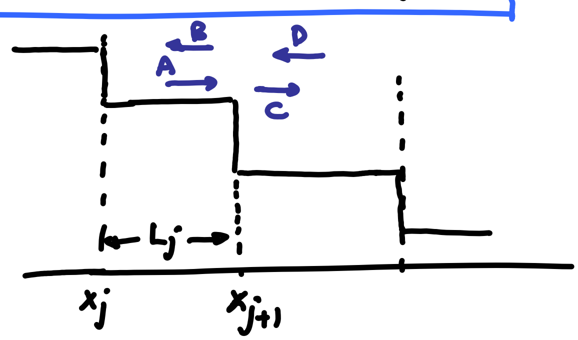
$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ \dfrac{k_{j+1}}{k_j} & -\dfrac{k_{j+1}}{k_j} \end{bmatrix}\begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}$$

Recall: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-1-1}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \dfrac{k_{j+1}}{k_j} & -\dfrac{k_{j+1}}{k_j} \end{bmatrix}}_{P_{j,\text{step}}}\begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}$$

$$\boxed{\hat{P}_{j,\text{step}} = \frac{1}{2}\begin{bmatrix} 1+\dfrac{k_{j+1}}{k_j} & 1-\dfrac{k_{j+1}}{k_j} \\ 1-\dfrac{k_{j+1}}{k_j} & 1+\dfrac{k_{j+1}}{k_j} \end{bmatrix}}$$



$$\psi_j(x_j) = A_j e^{ik_j x_j} + B_j e^{-ik_j x_j}$$

$$\psi_j(x_{j+1}) = A_j e^{ik_j(x_j+L_j)} + B_j e^{-ik_j(x_j+L_j)}$$

$$= \underbrace{A_j e^{ik_j L_j}}_{A_{j+1}} e^{ik_j x_j} + \underbrace{B_j e^{-ik_j L_j}}_{B_{j+1}} e^{-ik_j x_j}$$

$$\begin{cases} A_j e^{ik_j L_j} = A_{j+1} \\ B_j e^{-ik_j L_j} = B_{j+1} \end{cases} \Rightarrow \begin{bmatrix} e^{ik_j L_j} & 0 \\ 0 & e^{-ik_j L_j} \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix} = \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix}$$

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \underbrace{\begin{bmatrix} e^{ik_j L_j} & 0 \\ 0 & e^{-ik_j L_j} \end{bmatrix}^{-1}}_{\hat{P}_{j\,free}} \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix}$$

$$\boxed{P_{j,free} = \begin{bmatrix} e^{-ik_j L_j} & 0 \\ 0 & e^{ik_j L_j} \end{bmatrix}}$$

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \hat{P}_{j,free} \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix}$$

$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \hat{P}_{j,step} \begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}$$

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \underbrace{\hat{P}_{j,free}\, \hat{P}_{j,step}}_{\hat{P}_j} \begin{bmatrix} C_{j+1} \\ D_{j+1} \end{bmatrix}$$

$$\begin{matrix} A_j \\ B_j \end{matrix} \rightarrow \begin{matrix} A_{j+1} \\ B_{j+1} \end{matrix} \rightarrow \begin{matrix} C_{j+1} \\ D_{j+1} \end{matrix}$$

$$\boxed{\hat{P}_j = \frac{1}{2} \begin{bmatrix} \left(1 + \frac{k_{j+1}}{k_j}\right) e^{-ik_j L_j} & \left(1 - \frac{k_{j+1}}{k_j}\right) e^{-ik_j L_j} \\ \left(1 - \frac{k_{j+1}}{k_j}\right) e^{ik_j L_j} & \left(1 + \frac{k_{j+1}}{k_j}\right) e^{ik_j L_j} \end{bmatrix}}$$